

## BERGMAN COMPLEXES OF LATTICE PATH MATROIDS

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ABSTRACT. We give an explicit description of the poset of cells of Bergman complexes of Lattice Path Matroids and establish a criterion for simpliciality in terms of the shape of the bounding paths.

## 1. INTRODUCTION

The term *Bergman fan* has come to denote the polyhedral fan given as the logarithmic limit set of a complex algebraic subvariety  $V$  of  $\mathbb{C}^n$  and first introduced by George M. Bergman in [Ber71]. Sturmfels [Stu02] made the key observation that, if  $V$  is defined by linear equations, its Bergman fan can be constructed from the *matroid* associated to  $V$ . As Sturmfels' construction can be carried out also for nonrepresentable matroids, we get a Bergman fan associated to every matroid  $M$ . There is no loss of information in considering, instead of the whole fan, just its intersection with the unit sphere: the resulting spherical complex  $\Gamma_M$  is called the *Bergman complex* of the given matroid  $M$  (see Section 2.2 for the precise definitions).

The homotopy type of the complex  $\Gamma_M$  coincides with that of the order complex of  $\mathcal{L}(M)$ , the lattice of flats of  $M$ : this has been established by Ardila and Klivans [AK05], who proved that in fact the order complex of  $\mathcal{L}(M)$  subdivides  $\Gamma_M$ . This raises the question of the polyhedral structure of  $\Gamma_M$ . Here, progress was made by Feichtner and Sturmfels [FS05], who proved that  $\Gamma_M$  is subdivided by the nested set complex of  $M$ , a much coarser (simplicial) complex than the order complex of  $\mathcal{L}(M)$ . As an additional improvement, the second author [Dlu11] described a decomposition of the matroid types (see Section 2.1) associated to faces of the Bergman complex into connected direct summands.

To our knowledge, not only an effective way of computing the face structure of Bergman complexes is unknown, but even the question of whether the Bergman complex of a given matroid is simplicial can't be effectively answered in general.

*Lattice Path Matroids* were introduced by Bonin, de Mier and Noy [BdMN03] as a family of transversal matroids whose bases can be characterized by means of the lattice paths contained in the region of the plane bounded by two given lattice paths  $p$  and  $q$ . Lattice Path Matroids enjoy a host of nice enumerative and structural properties [BdM06], and they can be characterized among all matroids by a list of excluded minors [Bon10, Theorem 3.1]. Recently, Lattice Path Matroids appeared as a special case of the positroids used as indices of cells in Postnikov's stratification of the totally nonnegative Grassmannian [Pos06, Oh08] (for this 'special cells' the computation of the corresponding 'Grassmann Necklace' is particularly handy [Oh08, Section 6]).

In this paper we compute the polyhedral structure of the Bergman complex of a given Lattice Path Matroid, as well as a necessary and sufficient condition for  $\Gamma_M$  to be simplicial. In the geometric spirit of Lattice Path Matroids, our characterizations are in terms of the shapes of the bounding paths.

In Section 2 we start by reviewing the basic definitions and properties of Bergman complexes. In particular, we derive a description of the vertices of the Bergman complex of a Lattice Path Matroid in terms of *bays* and *land necks* of the bounding paths.

Section 3 contains our first main result, Theorem 3.1, where we characterize simpliciality of faces of the Bergman complex in terms of their vertex set - thus, again, in terms of bays and land necks. As a corollary we then obtain that the Bergman complex of a Lattice Path Matroid is simplicial if and only if any two vertically aligned bays determine a land neck.

We end with Section 4, in which we introduce a poset (again defined in terms of ‘chains of bays’ and some compatible sets of non – land necks, see Definition 4.2) that turns out to be isomorphic to the face poset of the Bergman complex (Theorem 4.3).

## 2. PRELIMINARIES

**2.1. Matroid Polytopes.** Let  $M$  be a matroid of rank  $d$  on the ground set  $[n] := \{1, \dots, n\}$ . By  $\mathcal{B}(M)$  we will denote the set of its bases,  $\mathcal{L}(M)$  the set of its flats. Given  $A \subseteq [n]$  we will write  $M/A$  for the *contraction* of  $A$  and  $M[A]$  for the restriction to  $A$  (for the basics on matroids we refer to [Oxl11]).

For every  $B \in \mathcal{B}(M)$  we consider a vector  $v(B) \in \mathbb{R}^n$  defined by  $v(B)_i = 1$  if  $i \in B$ ,  $v(B)_i = 0$  else.

The *matroid polytope* of  $M$  is the convex hull

$$P_M := \text{conv}\{v(B) \mid B \in \mathcal{B}(M)\}.$$

This is a polytope of dimension  $n - c(M)$ , where  $c(M)$  is the number of connected components of  $M$ , and is a subset of the hypersimplex  $\Delta_r = \text{conv}\{re_i \mid i = 1, \dots, n\}$ . It can be readily seen that for two matroids  $M_1, M_2$  we have  $P_{M_1 \oplus M_2} = P_{M_1} \times P_{M_2}$ . Without loss of generality we will restrict ourselves to the consideration of connected matroids, where  $P_M$  is  $(n - 1)$ -dimensional.

Given a flat  $F \in \mathcal{L}(M)$ , define the halfspace

$$H_F^+ := \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \leq \text{rank}(F)\}$$

and let  $H_F$  be the hyperplane bounding  $H_F^+$ . According to [FS05], we have

$$P_M = \Delta_r \cap \bigcap_{F \in \mathcal{L}(M)} H_F^+.$$

Let us consider the poset  $\mathcal{F}(M)$  of faces (closed cells) of  $P_M$  ordered by inclusion (see [Zie95, Definition 2.6]). Every  $f \in \mathcal{F}(M)$  is the matroid polytope of a minor  $M_f$  of  $M$ , called “the matroid type” of the face  $f$ , with set of bases

$$\mathcal{B}(M_f) = \{B \in \mathcal{B}(M) \mid v(B) \in V(f)\},$$

where  $V(f)$  denotes the set of vertices of  $f$ .

A maximal element  $f \in \mathcal{F}(M)$  (a *facet* of  $P_M$ ) can be of one of two types:

- (i)  $f$  lies on the boundary of  $\Delta_r$ ,
- (ii)  $f$  meets the interior of  $\Delta_r$ .

*Remark 1.*

- (i) If  $f$  is of type (i), then there is  $j$  such that  $x_j = 0$  for all  $x \in f$ . In particular this is true for  $x$  a vertex of  $f$ . This means that  $j$  is not contained in any basis of  $M_f$  - i.e.,  $j$  is a loop of  $M_f$ . Conversely, if  $j$  is a loop of  $M_f$ , then  $f$  is of type (i).

- (ii) The facet  $f$  is of type (ii) exactly when  $f \subseteq H_F$  for an  $F \in \mathcal{L}(M)$  such that  $M/F$  and  $M[F]$  are both connected. Such an  $F$  is called (following [FS05]) a *facet* of the matroid  $M$ , and is uniquely determined by  $f$ . We will write  $F_f$  for the facet corresponding to the type-(ii) facet  $f$ .

*Remark 2.* The vertices of a facet  $f$  of type (ii) are indexed by the bases of  $M_f$ , i.e., by the set

$$\mathcal{B}(M_f) = \{B \in \mathcal{B}(M) \mid |B \cap F_f| = \text{rank}(F_f)\}$$

**2.2. Bergman complexes.** Let  $M$  be a connected matroid of rank  $d$  on the ground set  $[n]$ .

We consider the polar dual  $P_M^\vee$  of  $P_M$  (see e.g. [Zie95, Definition 2.10]) and let  $\mathcal{F}^\vee(M)$  denote its poset of faces. Duality determines canonical order-reversing bijections  $\mathcal{F}(M) \xrightarrow{\sim} \mathcal{F}^\vee(M) \xrightarrow{\sim} \mathcal{F}(M)$ . In particular, every face  $\alpha \in \mathcal{F}^\vee(M)$  has a matroid type  $M_\alpha := M_{\alpha^\vee}$ .

**Definition 2.1.** The *Bergman complex* of  $M$  is the polyhedral subcomplex  $\Gamma_M$  of  $P_M^\vee$  with set of faces

$$\Gamma_M := \{\alpha \in \mathcal{F}^\vee(M) \mid M_\alpha \text{ loopfree}\},$$

which we regard as a downwards closed subposet of  $\mathcal{F}^\vee(M)$ .

*Remark 3.* According to Remark 1, the vertices of the Bergman complex are exactly the faces of the form  $f^\vee$  where  $f$  is a facet of type (ii).

Let  $\gamma_1, \dots, \gamma_k$  be the vertices of a face  $\alpha \in \Gamma_M$ . The face  $\alpha^\vee$  of  $P_M$  is the intersection of its adjacent facets  $\gamma_1^\vee, \dots, \gamma_k^\vee$ .

Therefore  $v(\mathcal{B}(M_\alpha)) = V(\alpha^\vee) = \bigcap_{i=1}^k V(\gamma_i^\vee) = \bigcap_{i=1}^k v(\mathcal{B}(M_{\gamma_i}))$ , and with Remark 2 we can write

$$\mathcal{B}(M_\alpha) = \{B \in \mathcal{B}(M) \mid |B \cap F_{\gamma_i^\vee}| = \text{rank}(F_{\gamma_i^\vee}) \text{ for } i = 1, \dots, k\}$$

The face  $\alpha \in \Gamma_M$  is simplicial if every proper subset of its vertices determines a proper subface. Conversely,  $\alpha$  fails to be simplicial iff

$$\mathcal{B}(M_\alpha) = \{B \in \mathcal{B}(M) \mid |B \cap F_{\gamma^\vee}| = \text{rank } F_{\gamma^\vee} \text{ for } \gamma \in U\}$$

for a proper subset  $U \subsetneq V(\alpha)$ .

**2.3. Lattice Path matroids.** Let  $p, q$  be lattice paths in the plane with common begin- (say at the origin  $(0, 0)$ ) and endpoint (say at a point  $(m, r)$ ). We will assume that  $p$  never goes below  $q$ . We will write  $p$  and  $q$  as words

$$p = p_1 \dots p_{m+r} \quad q = q_1 \dots q_{m+r}$$

where each ‘letter’ is  $N$  or  $E$ , signaling a step ‘North’  $(0, 1)$  or ‘East’  $(1, 0)$ .

By  $[p, q]$  we will denote the set of lattice paths from  $(0, 0)$  to  $(m, r)$  that never go above  $p$  or below  $q$ .

For any  $s \in [p, q]$  write  $s = s_1 \dots s_{m+r}$  and define

$$B(s) := \{i \mid s_i = N\}.$$

**Lemma 2.2** ([BdM06]). *The set  $\{B(s) \mid s \in [p, q]\}$  is the set of bases of a matroid  $M(p, q)$  on the ground set  $[m + r]$ .*

**Definition 2.3.** A *Lattice Path Matroid (LPM)* is any matroid of the form  $M(p, q)$  for two lattice paths  $p, q$  as above.

A Lattice Path Matroid  $M(p, q)$  is connected iff the paths  $p$  and  $q$  never touch except at  $(0, 0)$  and  $(m, r)$  [BdM06, Theorem 3.5].

**Assumption.** Unless otherwise stated, in the following we will consider connected Lattice Path Matroids.

**Definition 2.4** (Fundamental flats, bays, land necks). We will say that a lattice point  $(y_1, y_2)$  on the upper path  $p$  is a *bay* of  $p$  if  $p_{y_1+y_2}p_{y_1+y_2+1} = EN$ . Similarly, a point  $(z_1, z_2)$  on  $q$  is a bay for  $q$  if  $q_{z_1+z_2}q_{z_1+z_2+1} = NE$ . Let  $U_p$ , resp.  $U_q$  be the set of bays of  $p$ , resp. of  $q$ .

The *fundamental flats* of  $M$  are sets of the form  $\{1, \dots, y_1+y_2\}$  for an  $(y_1, y_2) \in U_p$  or of the form  $\{z_1 + z_1 + 1, \dots, m + r\}$  for  $(z_1 + z_2) \in U_q$ .

We will say that  $i \in [m + r]$  is a *land neck* of  $M(p, q)$  if the endpoint of  $p_1 \dots p_{i+1}$  lies one unit North of the endpoint of  $q_1 \dots q_i$ . The set of land necks is  $S(p, q)$ .

The following is a rephrasing of Corollary 2.14 of [Bon10] and Theorem 5.3 of [BdM06].

**Lemma 2.5.** *The flacets of a connected Lattice Path Matroid  $M(p, q)$  are*

- (a) *the fundamental flats,*
- (b) *the singletons  $\{i\} \subseteq [m + r] \setminus S(p, q)$ .*

*Example 2.6.* Consider the lattice path matroid of rank 8 on  $[1, \dots, 16]$ . Its lattice presentation and some typical flacets are given on the left of Figure 1. The fundamental flats shown correspond to the p-bays  $(3, 3)$ ,  $(4, 6)$  and the q-bay  $(2, 1)$ . The only singleton is  $\{8\}$ . On the right side there is a lattice path presentation of the matroid type whose flacets correspond to the given flacets. Remark that in order to get such a presentation of the minor, one has to change the initial order of the ground set.

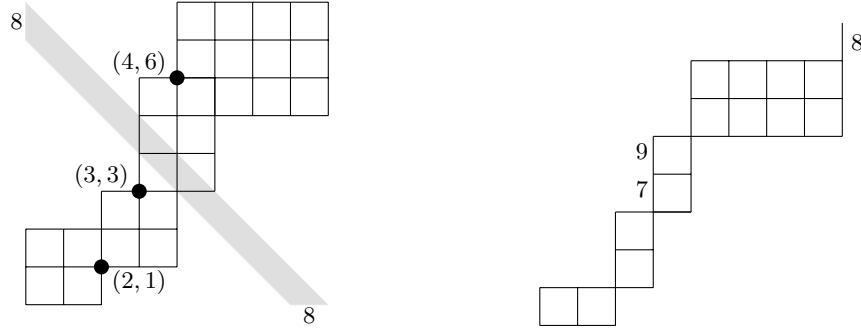


FIGURE 1. Some flacets and a lattice representation of the matroid type whose dual has those flacets as vertices

### 3. A SIMPLICIALITY CRITERION

The goal of this section is to give a complete characterization of which Lattice Path Matroids possess a simplicial Bergman complex.

Let  $M = M(p, q)$  be a connected Lattice Path Matroid, and  $\alpha$  a face of its Bergman complex.

*Remark 4* (Notation). Given  $B \in \mathcal{B}(M)$  we will write  $p(B)$  for the corresponding lattice path. A *node* of the matroid type  $M_\alpha$  is an integer point that is visited by every path  $p(B)$  with  $B \in \mathcal{B}(M_\alpha)$ .

*Remark 5.* Let  $F$  be a flacet of  $M$ , and  $B \in \mathcal{B}(M)$ . Then  $|B \cap F| = \text{rank}(F)$  if and only if one of the following holds

- (a)  $F$  is a fundamental flat and  $p(B)$  goes through the corresponding corner,

(b)  $F = \{i\}$  and  $p(B)_i = N$ .

We will then say that the path  $p(B)$  satisfies the constraint imposed by  $F$ .

**Theorem 3.1.** *Let  $\alpha$  be a face of the Bergman complex of a connected Lattice Path Matroid  $M(p, q)$ . Then  $\alpha$  is simplicial unless the facets corresponding to its vertices include*

- (1) *a fundamental flat  $F$  corresponding to a bay  $(x, z)$  of  $p$  and*
- (2) *a fundamental flat  $G$  corresponding to a bay  $(x, y)$  of  $q$*

*with  $z - y > 1$ .*

*Proof.* Let the vertices of  $\alpha$  be as in the claim. Then, for all  $B \in \mathcal{B}(M_\alpha)$ , the lattice path  $p(B)$  passes through  $(x, y)$  and  $(x, z)$ . So it satisfies  $p(B)_j = N$  for  $j = x + y + 1, \dots, x + z$ . The converse is also true: if all lattice paths  $p(B)$  with  $B \in \mathcal{B}(M_\alpha)$  satisfy  $p(B)_j = N$  for  $j = x + y + 1, \dots, x + z$ , then every  $p(B)$  must pass both  $(x, y)$  and  $(x, z)$ . We conclude that  $F$  and  $G$  correspond to vertices of  $V(\alpha)$  iff  $\{x + y + 1, \dots, x + z\}$  correspond to vertices of  $V(\alpha)$ . In view of Remark 2.2, we see that  $\alpha$  is not simplicial.

For the reverse implication, suppose now  $\alpha$  not to be simplicial. Again by Remark 2.2 there is  $\gamma_0 \in V(\alpha)$  such that

$$(1) \quad \mathcal{B}(M_\alpha) = \{B \in \mathcal{B}(M) \mid |B \cap F_\gamma| = \text{rank}(F_\gamma) \text{ for } \gamma \in V(\alpha) \setminus \{\gamma_0\}\}.$$

Case 1:  $F_{\gamma_0}$  is a fundamental flat. W.l.o.g.  $F_{\gamma_0}$  corresponds to a  $p$ -bay  $(x, z)$ .

*Claim:* Let  $u$  be minimal such that  $(x, u)$  is a node of  $M_\alpha$ . Then  $(x, u)$  lies on  $q$ .

Let  $v$  be maximal such that  $(x, v)$  is a node of  $M_\alpha$ . There is a  $M_\alpha$ -path

$$s = s_1 \dots s_{x+u-1} N \dots N E s_{x+v+2} \dots s_{r+m}.$$

Now consider the path

$$s' = s_1 \dots s_{x+u-1} E N \dots N s_{x+v+2} \dots s_{r+m}.$$

If  $s' = p(B)$  for a basis  $B \in \mathcal{B}(M)$ , then it would satisfy the constraints given by  $F_\gamma$  for  $\gamma \in V(\alpha) \setminus \gamma_0$  but not the one given by  $F_{\gamma_0}$  (it does not pass through the  $p$ -bay  $(z, x)$ ). Thus, it would be an element in the right hand side but not on the left hand side of Equation 1: a contradiction.

We conclude that  $s' \notin [p, q]$ , meaning that the point  $(x + 1, u - 1)$  is not a lattice point between  $q$  and  $p$ , i.e.,  $(x, u)$  lies on  $q$ .

The Claim shows that there is a node  $(x, u)$  of  $M_\alpha$  which lies on  $q$ . In particular, there has to be a  $q$ -bay  $(x, y)$  with  $u \leq y \leq z$  which is also a node of  $M_\alpha$ , q.d.e.  $\triangle$

Case 2:  $F_{\gamma_0}$  is a singleton  $\{i'\}$ . Then, every path  $s$  of  $M_\alpha$  has  $s_{i'} = N$ .

*Claim:*  $M_\alpha$  has a node  $(x_1, x_2)$  with  $x_1 + x_2 = i'$ .

To see this, consider two paths  $s, s'$  of  $M_\alpha$  such that  $s_1 \dots s_{i'}$  ends at  $(y_1, y_2)$  and  $s'_1 \dots s'_{i'}$  ends at  $(y'_1, y'_2)$  with  $y'_2 \geq y_2$ . We want to prove that  $y'_2 = y_2$ .

By way of contradiction suppose  $y'_2 - y_2 > 0$  and let  $(a_1, a_2), (b_1, b_2)$  with  $a_1 + a_2 < i' < b_1 + b_2$  be on both  $s, s'$  and such that the path  $s_{a_1+a_2+1} \dots s_{b_1+b_2}$  is always south of  $s'_{a_1+a_2+1} \dots s'_{b_1+b_2}$ . In particular,  $s'_{a_1+a_2} = N$  and  $s'_{b_1+b_2-1} = E$  and there are no nodes of  $M_\alpha$  between  $(a_1, a_2)$  and  $(b_1, b_2)$ . The path

$$s'_1 \dots s'_{a_1+a_2} N s_{a_1+a_2+2} \dots s_{b_1+b_2-1} E s_{b_1+b_2+1} \dots s_{r+m}$$

is thus a path of  $M_\alpha$  that after  $i'$  steps ends at a point  $(y''_1, y''_2)$  with  $y''_2 = y_2 + 1$ .

By repeating this operation we can assume w.l.o.g. that  $y'_2 - y_2 = 1$ . Now, in this case the path

$$s'_1 \dots s'_{i'-1} E s_{i'+1} \dots s_{r+m}$$

is an element of the right-hand side but not on the left-hand side of (1): a contradiction.  $\triangle$

Thus, there is  $x$  such that for all  $B \in \mathcal{B}(M_\alpha)$  the path  $p(B)_1 \dots p(B)_{i'}$  ends at  $(x, i' - x)$ . Since  $F_{\gamma_0} = \{i'\}$ , we have that both  $(x, i' - x - 1)$  and  $(x, i' - x)$  are nodes of  $M_\alpha$ .

To prove the theorem it now suffices to prove the following claim.

*Claim: There is a  $p$ -bay  $(x, z)$  north and a  $q$ -bay  $(x, y)$  south of  $(x, i' - x)$ .*

Let  $(x, u)$  be the lowest node of  $M_\alpha$  south of  $(x, i' - x - 1)$ . Then there is a path  $s$  of  $M_\alpha$  with  $s_{x+u} = E$ . Similarly let  $v$  be maximal s.t.  $(x, v)$  is a node of  $M_\alpha$  and let  $t$  be a path of  $M_\alpha$  with  $t_{x+v+1} = E$ . Consider the paths

$$\begin{aligned} s' &:= s_1 \dots s_{x+u-1} N \dots N E s_{i'+1} \dots s_{r+m}, \\ t' &:= t_1 \dots t_{i'-1} E N \dots N t_{x+v+2} \dots t_{r+m}. \end{aligned}$$

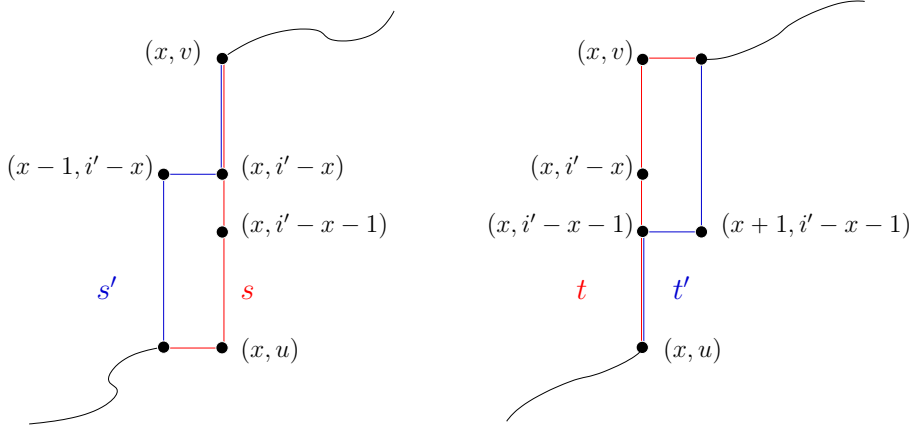


FIGURE 2. The paths  $s$ ,  $t$ ,  $s'$  and  $s$  in the proof of the claim.

Neither  $s'$  nor  $t'$  can be a path of  $M_\alpha$ , because they violate the constraint of  $F_{\gamma_0}$  by  $s'_{i'} = t'_{i'} = E$ .

Therefore, in order to uphold Equation (1), each of the paths  $s'$  and  $t'$ , if at all an element of  $[p, q]$ , must violate some of the constraints given by the  $F_\gamma$  with  $\gamma \in V(\alpha) \setminus \{\gamma_0\}$ .

We conclude that for  $s'$  one of the following must occur:

- (si) the point  $(x, i' - x - 1)$  lies on  $p$  (since  $s'$  passes  $(x - 1, i' - x)$ ), or
- (sii) there is  $\gamma \in V(\alpha)$  such that  $F_\gamma$  corresponds to a  $q$ -bay  $(x, y)$  between  $(x, i' - x)$  and  $(x, u)$ .

Similarly, for  $t'$  not to be a path of  $M_\alpha$  one of the following must enter:

- (ti) the point  $(x, i' - x)$  lies on  $q$  (since  $t'$  passes  $(x + 1, i' - x - 1)$ ), or
- (tii) there is  $\gamma \in V(\alpha)$  such that  $F_\gamma$  corresponds to a  $p$ -bay  $(x, z)$  between  $(x, i' - x)$  and  $(x, v)$ .

Now, if both (si) and (ti) were true,  $i$  would be a land neck; on the other hand, (si) and (tii) together imply that there is a point on  $p$  south of a  $p$ -bay, and similarly from (ti) and (sii) follows the existence of a point on  $q$  that north of a  $q$ -bay. Thus, none of the above cases can enter.

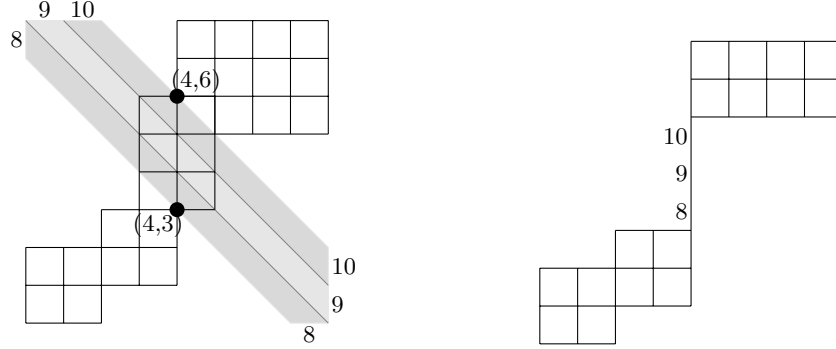


FIGURE 3. Some facets and a lattice representation of the matroid type whose dual has those facets as vertices

We conclude that both (sii) and (tii) must hold, proving the claim.  $\triangle$   
 $\square$

**Corollary 3.2.** *The Bergman fan of a connected lattice path matroid  $M(p, q)$  is simplicial iff every pair of vertically aligned bays is a land neck (I.e., if  $(x_1, x_2) \in U_p$  and  $(x_1, x'_2) \in U_q$ , then  $x_2 - x'_2 = 1$ .)*

*Example 3.3.* Consider the lattice paths matroid given in Figure 3. The face  $\alpha$  whose vertices correspond to the bays  $(4, 3)$ ,  $(4, 6)$  and the singletons  $8, 9, 10$  in Figure 3 is a minimal non-simplicial face.

#### 4. COMBINATORIAL STRUCTURE

**4.1. The poset of faces.** Let  $M = M(p, q)$  be a lattice path matroid with set of bays  $U$  and set of land necks  $S(p, q)$ .

Define a partial order on  $U$  by setting

$$(x_1, x_2) < (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 < y_2.$$

We will denote with  $\Delta(U)$  the set of chains (i.e., totally ordered subsets) of  $U$ , ordered by inclusion.

Every chain

$$\omega := \{(a_1, b_1) < (a_2, b_2) < \dots\}$$

of  $U$  defines a partition

$$\pi(\omega) = \pi_1(\omega) \uplus \dots \uplus \pi_{|\omega|+1}(\omega)$$

of the set  $[m + r]$  with  $j$ -th block  $\pi_j(\omega) := \{a_{j-1} + b_{j-1} + 1, \dots, a_j + b_j\}$  for  $j = 1, \dots, |\omega| + 1$ , where we set  $(a_0, b_0) = 0$  and  $(a_{|\omega|+1}, b_{|\omega|+1}) = (m, r)$ .

**Definition 4.1.** The chain  $\omega$  defines a minor

$$M(\omega) = M_1(\omega) \oplus \dots \oplus M_{|\omega|+1}(\omega)$$

of  $M$ , where  $M_j(\omega)$  is the lattice path matroid represented by the lattice paths from  $(a_{j-1}, b_{j-1})$  to  $(a_j, b_j)$  lying between  $p$  and  $q$ .

*Remark 6.* A closed expression for  $M_i(\omega)$  can be given as follows. Let  $\pi_{<i}(\omega) := \bigcup_{j < i} \pi_j(\omega)$  resp.  $\pi_{>i}(\omega) := \bigcup_{j > i} \pi_j(\omega)$ . Then, Corollary 3.2 of [BdM06] and elementary duality consideration show that

$$M_i(\omega) = \begin{cases} M[\pi_i(\omega)] & \text{if } (a_i, b_i) \in U_q, (a_{i+1}, b_{i+1}) \in U_p \\ M[\pi_{\leq i}(\omega)]/\pi_{<i}(\omega) & \text{if } (a_i, b_i) \in U_p, (a_{i+1}, b_{i+1}) \in U_p \\ M[\pi_{\geq i}(\omega)]/\pi_{>i}(\omega) & \text{if } (a_i, b_i) \in U_q, (a_{i+1}, b_{i+1}) \in U_q \\ M/(\pi_{<i}(\omega) \cup \pi_{>i}(\omega)) & \text{if } (a_i, b_i) \in U_p, (a_{i+1}, b_{i+1}) \in U_q \end{cases}$$

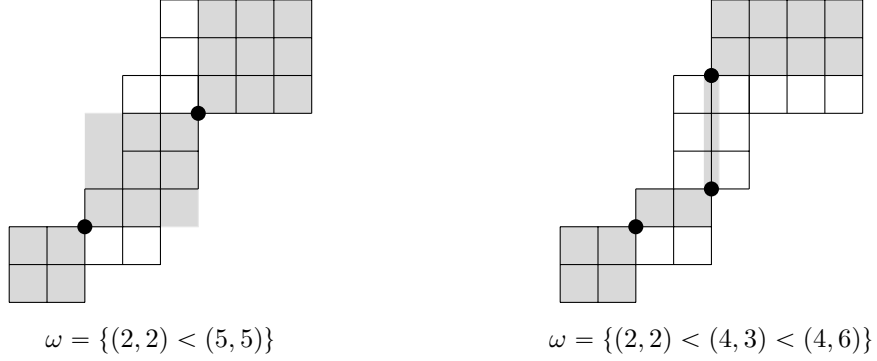


FIGURE 4. The direct summands for  $M(\omega)$  are the lattice path matroids that appear in the shaded regions.

*Remark 7.* Notice that, for  $\omega \in \Delta(U)$ , every  $M_j(\omega)$  is connected and loopfree. This direct sum decomposition is a special case of the decomposition introduced in [Dlu11].

Let the set of all sets of non-land necks  $\mathcal{I} := \mathcal{P}([m+r]) \setminus S(p,q)$  be partially ordered by inclusion.

**Definition 4.2.** Given a lattice path matroid  $M$  recall the above notations and define  $\mathcal{Q}(p,q)$  as the subposet of  $\Delta(U) \times \mathcal{I}$  defined by

$$\left\{ (\omega, J) \in \Delta(U) \times \mathcal{I} \left| \begin{array}{ll} \pi_i(\omega) \subseteq J & \text{if } a_{i-1} = a_i; \\ \pi_i(\omega) \cap J \text{ is a flat of } M_i(\omega) & \text{else.} \end{array} \right. \right\}$$

*Remark 8.* In view of Remark 6, the second condition in Definition 4.2 can be replaced by

$$\left\{ \begin{array}{ll} J \cap \pi_i(\omega) \text{ is a flat of } M & \text{if } (a_i, b_i) \in U_q, (a_{i+1}, b_{i+1}) \in U_p, \\ (J \cap \pi_i(\omega)) \cup \pi_{<i}(\omega) \text{ is a flat of } M & \text{if } (a_i, b_i) \in U_p, (a_{i+1}, b_{i+1}) \in U_p, \\ (J \cap \pi_i(\omega)) \cup \pi_{>i}(\omega) \text{ is a flat of } M & \text{if } (a_i, b_i) \in U_q, (a_{i+1}, b_{i+1}) \in U_q, \\ J \cup \pi_{<i}(\omega) \cup \pi_{>i}(\omega) \text{ is a flat of } M & \text{if } (a_i, b_i) \in U_p, (a_{i+1}, b_{i+1}) \in U_q. \end{array} \right.$$

**Theorem 4.3.** Let  $M(p,q)$  be a connected lattice path matroid. Then there is a poset isomorphism

$$\Gamma_{M(p,q)} \simeq \mathcal{Q}(p,q)$$

*Proof.* Faces of  $\mathcal{B}(M)$  are cells of  $P_M^\vee$ , and thus uniquely determined by their vertices. Therefore, we can identify  $\Gamma_M$  and  $V(\Gamma_M)$ .

For a face  $\alpha$  of the Bergman complex of  $M$  the following hold:

- (a)  $V(\alpha) \cap U \in \Delta(U)$ .

To prove this, consider two points  $(a,b), (c,d) \in V(\alpha) \cap U$  such that  $a \leq c$  and  $b \geq d$ . Every lattice path  $p(B)$  for  $B \in M_\alpha$  must pass through both points, therefore  $a \leq c$  implies  $b \leq d$ , so  $b = d$ . Now we have  $p(B)_j = E$  for all  $B \in \mathcal{B}(M_\alpha)$  and every  $j \in \{a+b+1, \dots, c+b\}$  - but since  $M_\alpha$  by definition must be loopless, there can't be any such  $j$ , thus  $a = c$  and we are done.

Write  $\omega = (a_1, b_1) < \dots < (a_k, b_k)$  for the chain  $V(\alpha) \cap U$ , and let  $i \in [k]$ .

- (b) If  $a_{i-1} = a_i$ , then  $V(\alpha) \cap \pi_i(\omega) \subseteq \mathcal{I}$ .

In fact, in this case we are in the situation of Theorem 3.1, and every  $\{a_{i-1} + b_{i-1} + k\}$  for  $k = 1, \dots, b_i - b_{i-1}$  is a vertex of  $\alpha$  (in particular, not a land neck).



- (c) If  $a_{i-1} < a_i$ , then  $\pi_i(\omega) \cap \mathcal{I}$  is a flat of  $M$ .  
 Namely, in this case the lattice paths  $p(B)_{a_{i-1}+b_{i-1}+1} \cdots p(B)_{a_i+b_i}$ , for  $B \in \mathcal{B}(M_\alpha)$ , determine a lattice path matroid  $M'$  which is a direct summand of  $M$ , hence loopfree. Moreover,  $M'$  is isomorphic to the contraction  $M_i(\omega)/(J \cap \pi_i(\omega))$ . By [Oxl11, Exercise 3.1.8], this contraction is loopfree if and only if  $J \cap \pi_i(\omega)$  is a flat of  $M_i(\omega)$ .

In view of (a), (b), (c) above, the following function is well defined

$$\psi : \Gamma_M \rightarrow \mathcal{Q}(p, q); \quad X \rightarrow (X \cap U, X \cap \mathcal{I})$$

and clearly order-preserving. Moreover, it admits an order-preserving inverse given by

$$\mathcal{Q}(p, q) \rightarrow \Gamma_M; \quad (\omega, J) \rightarrow \omega \cup J,$$

is easily seen to be well-defined, as it sends  $(\omega, J)$  to the vertex set of the face  $\alpha$  with

$$M_\alpha = M(\omega)/J = \bigoplus_{a_{i-1} \neq a_i} M_i(\omega)/(J \cap \pi_i(\omega)) \bigoplus_{a_{i-1} = a_i} M_i(\omega)$$

which is loopfree because all its direct summands are.  $\square$

**4.2. Polyhedral structure of faces.** As a face of  $P^\vee$ , every face of  $\Gamma_M$  is the convex hull of its vertices. We would like to characterize such polyhedra.

*Notation 1.* By Remark 1 and Remark 3 the vertices of the Bergman complex correspond to flacets of the matroid. In the case of lattice path matroids, by Lemma 2.5 the flacets are fundamental flats - which, in turn, correspond to bays - and non-land neck singletons.

Thus, for a subset  $X \subset U \cup ([m+r] \setminus S(p, q))$  let  $\gamma(X)$  denote the set of vertices  $v$  of  $P^\vee$  for which  $F_v \in X$  either is an element of  $X \setminus U$  or correspond to a bay in  $X \cap U$ .

Given a set of points  $A$  in general position let  $\Delta_A = \text{conv}(A)$  denote the simplex on the vertex set  $A$ . Moreover, let  $\diamond_A$  be the polytope obtained as the suspension of  $\Delta_A$ .

*Remark 9.* Recall that the poset of faces of the *join*  $P * Q$  of two polytopes  $P$  and  $Q$  is obtained from the face posets  $\widehat{\mathcal{F}}(P)$  and  $\widehat{\mathcal{F}}(Q)$  (these are the face posets of  $P$  and  $Q$ , each with an added unique smallest element  $0_P$  resp.  $0_Q$ ) by

$$\mathcal{F}(P * Q) \simeq \widehat{\mathcal{F}}(P) \times \widehat{\mathcal{F}}(Q)$$

**Corollary 4.4.** *The face of  $\Gamma_M$  corresponding to  $(\omega, J)$  is a join*

$$P(\omega, J) := \Delta_{\gamma(A)} *_{i: a_{i-1} < a_i} \Delta_{\gamma(J \cap \pi_i(\omega))} *_{i: a_{i-1} = a_i} \diamond_{\gamma(\pi_i(\omega))}$$

where  $A := \{(a_j, b_j) \in \omega \mid a_{j-1} < a_j < a_{j+1}\}$  is the set of all bays in  $\omega$  that do not bound a land neck in  $M(\omega)$ .

*Remark 10.* Notice that if there is no  $i$  with  $a_{i-1} = a_i$ , then every term of the above join is a simplex, and thus  $P(\omega, J)$  is a simplex, in agreement with the condition found in Theorem 3.1.

*Proof.* Since the vertex set of  $P(\omega, J)$  is precisely  $\gamma(\omega \cup J)$ , it is enough to prove isomorphism of face posets.

Given a set  $X$  and distinct elements  $y_1, y_2 \notin X$ , write  $B_X$  for the poset of all subsets of  $X$ ,  $C_{X, y_1, y_2}$  for the poset of all subsets  $Y \subseteq X \cup \{y_1, y_2\}$  such that

$\{y_1, y_2\} \subseteq Y$  if and only if  $X \subseteq Y$ . Then  $B_X \simeq \widehat{\mathcal{F}}(\Delta_X)$  and  $C_{X, y_1, y_2} \simeq \widehat{\mathcal{F}}(\diamond_X)$  and the face poset of  $P(\omega, J)$  is isomorphic to

$$B_A \times \prod_{a_{i-1} \neq a_i} B_{J \cap \pi_i(\omega)} \prod_{a_{i-1} = a_i} C_{\pi_i(\omega), (a_{i-1}, b_{i-1}), (a_i, b_i)}.$$

Now to prove that the map

$$\begin{aligned} \mathcal{Q}(p, q)_{\leq (\omega, J)} &\rightarrow \mathcal{F}(P(\omega, J)), \\ (\omega', J') &\mapsto (\omega' \cap A, \underbrace{J' \cap \pi_i(\omega), \dots, (\{a_{i-1}, a_i\} \cap \omega')}_{a_{i-1} < a_i} \cup \underbrace{(J' \cap \pi_i(\omega))}_{a_{i-1} = a_i}, \dots) \end{aligned}$$

is a poset isomorphism amounts to a routine check in view of Theorem 3.1 and of the fact that, for every  $(\omega, J) \in \mathcal{Q}(p, q)$ ,  $(\omega, J') \in \mathcal{Q}(p, q)$  for all  $J' \subseteq J$  (since loop-freeness of the quotients  $M_i(\omega)/J' \cap \pi_i(\omega)$  is implied by loop-freeness of  $M_i(\omega)/J \cap \pi_i(\omega)$ , see [BdM06, Corollary 3.2])  $\square$

*Example 4.5.* Consider the lattice given in Figure 3 on page 7. The face  $\alpha$  whose vertices correspond to the bays  $(4, 3)$ ,  $(4, 6)$  and the singletons 8, 9, 10 in Figure 3 is a triangular bipyramid obtained by suspending a 2-simplex (whose vertices correspond to the singletons 8, 9, 10) between two additional vertices (corresponding to the bays  $(4, 3)$  and  $(4, 6)$ ).

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